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A RATIONAL HOMOLOGY DISK SMOOTHING OF $W_{p,q,r}$

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Abstract. We prove that a rational homology disk smoothing of the singularity $W_{p,q,r}$ is $\mathbb{Q}$-Gorenstein by using its sandwiched structure.

1. Introduction

The concept of $\mathbb{Q}$-Gorensteiness plays a crucial role in the study of the Kollár–Shepherd-Barron–Alexeev compactifications of the moduli spaces of complex surfaces of general type. We say that a smoothing $\mathcal{Z} \to \Delta$ of a normal surface singularity Z is $\mathbb{Q}$-*Gorenstein* if some multiple of the canonical class of $\mathcal{Z}$ is Cartier. Originally, Wahl [6] and Kollár–Shepherd-Barron [3] classified quotient surface singularities that allow $\mathbb{Q}$-Gorenstein smoothings. These are cyclic quotient surface singularities of the form $\frac{1}{dn^2}(1, dna-1)$, where integers $d \geq 1$, $n > a \geq 1$, and $(n, a) = 1$. Such singularities are referred to as *singularities of class* T. More recently, Bhupal–Stipsicz [1] classified the resolution graphs of weighted homogeneous surface singularities that admit smoothings with the rational homology of the 4-disk.

One of the earliest examples that can be smoothed with rational homology disks is the singularity $W_{p,q,r}$, initially discovered by Wahl [6]. Here, $W_{p,q,r}$ is a singularity classified by Bhupal–Stipsicz [1], and its dual graph is illustrated in Figure 1.

In this paper, we prove that a rational homology disk smoothing of $W_{p,q,r}$ is indeed $\mathbb{Q}$-Gorenstein. It's worth noting that Wahl [7] had already established the $\mathbb{Q}$-Gorenstein property of rational homology disk smoothings of all the singularities given in Bhupal–Stipsicz [1]. However,

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in this paper, we present a different, simpler method to prove its $\mathbb{Q}$-Gorensteiness.

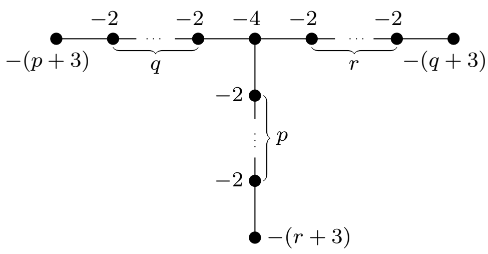

Figure 1. $W(p, q, r)$, where $p, q, r \geq 0$

Theorem 1.1. *The smoothing of $W_{p,q,r}$ whose Milnor fiber is a rational homology disk is $\mathbb{Q}$-Gorenstein.*

We briefly sketch the idea of the proof. We first show that $W_{p,q,r}$ is a sandwiched surface singularity; See Section 2 for details. Then one can apply the $\mathbb{Q}$-Gorensteinness criterion given by de Jong–van Straten [2]. In details, for each smoothing of a sandwiched surface singularity, there is a certain matrix, called an *incidence matrix*, that corresponds to the given smoothing. de Jong–van Straten [2] showed that a smoothing of a sandwiched surface singularity is $\mathbb{Q}$-Gorenstein if and only if its incidence matrix satisfies a certain simple condition; Proposition 3.1. So we first find the incidence matrix corresponding to the rational homology disk smoothing of $W_{p,q,r}$ and we then show that the incidence matrix satisfies the criterion.

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2. Sandwiched surface singularities

A *sandwiched surface singularity* $(X, 0)$ is a normal surface singularity admitting a birational morphism to $\rho\colon (X, 0) \to (\mathbb{C}^2, 0)$. Sandwiched surface singularities are rational singularities characterized by their dual resolution graphs, so-called *sandwiched graphs*:

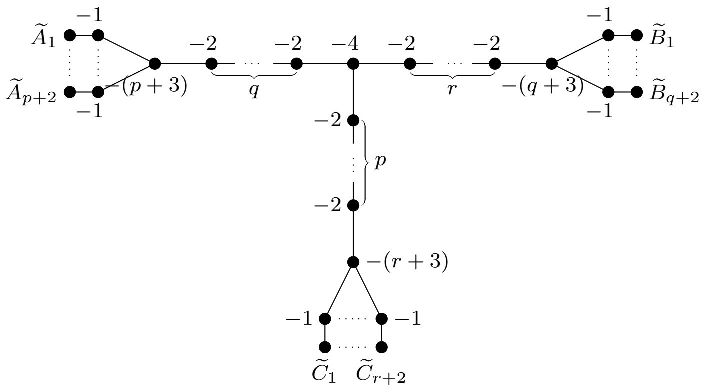

FIGURE 2. A sandwiched structure for $W_{p,q,r}$

DEFINITION 2.1 (Spivakovsky [5, Definition 1.9]). A graph Γ is called a *sandwiched graph* if it is the dual resolution graph of a rational surface singularity that can be blown down to a smooth point by adding new vertices with weights (-1) on the proper places.

PROPOSITION 2.2 (Spivakovsky [5, Proposition 1.11]). *A normal surface singularity is sandwiched if and only if its dual resolution graph is sandwiched.*

LEMMA 2.3. *The singularity $W_{p,q,r}$ is sandwiched.*

Proof. It is already proved in Park–Shin [4]. Its dual graph is sandwiched as given in Figure 2. □

2.1. Decorated curves

In their work on sandwiched surface singularities (X, p), de Jong and van Straten [2] introduced a pair (C, l). This pair consists of a plane curve singularity $C = \cup_{i=1}^{s} C_i$ and an assignment function $l\colon \{C_i \mid i = 1, \dots, e\} \to \mathbb{N}$ that characterizes the singularity X in terms of (C, l). Let's briefly revisit how one can derive (C, l) from (X, p). For a more in-depth explanation, please refer to de Jong-van Straten [2].

DEFINITION 2.4 (de Jong-van Straten [2, Definition 1.4]). A *decorated germ* is a pair (C, l), comprising a plane curve singularity $C = \cup_{i=1}^{s} C_i \subset \mathbb{C}^2$ passing through the origin and an assignment function $l\colon \{C_i \mid i = 1, \dots, s\} \to \mathbb{N}$. This assignment function must satisfy the condition $l(C_i) \geq m(C_i)$, where $m(C_i)$ denotes the sum of multiplicities of branch C_i in the multiplicity sequence of the minimal resolution of C.

Using a sandwiched graph structure, it is possible to construct a decorated curve (C, l) for (X, p). The dual graph of the minimal resolution (V, E) of (X, p) is, in fact, sandwiched and can be transformed into a smooth point by introducing some (-1)-vertices. Simultaneously, one can embed (V, E) into a blow-up space $(\widetilde{\mathbb{C}}^2, F)$ of $\mathbb{C}^2$ centered at the origin (including its infinitely near point), where F represents the set of exceptional divisors. For each (-1)-curve $F_i \in F$, one selects a *curvetta* $\widetilde{C}_i$ (essentially, a small segment of a curve) that intersects F_i transversely. The union of these $\widetilde{C}_i$ segments, denoted as $\widetilde{C}$, is then mapped to $C = \rho(\widetilde{C}) = \bigcup_{F_i \leq F} C_i$, where each C_i is the image of $\widetilde{C}_i$ under the map ρ. At this point, C can be regarded as a germ of plane curves passing through the origin 0. To decorate C_i, assign the number l_i, which represents the sum of multiplicities of the blowing-up points located on the strict transform of C_i.

A decorated curve for $W_{p,q,r}$ is given in Figure 2.

LEMMA 2.5. *For* $W_{p,q,r}$, $\widetilde{C} = (\cup \widetilde{A}_i) \bigcup (\cup \widetilde{B}_j) \bigcup (\cup \widetilde{C}_k)$ *and* $l(A_i) = q + 3$, $l(B_j) = r + 3$, $\quad l(C_k) = p + 3$.

For details, refer Park–Shin [4, §15.2].

2.2. Picture deformations

A one-parameter deformation of a sandwiched surface singularity (X, p) can be traced back to a one-parameter deformation of its decorated curve (C, l). The detailed deformation theory is outlined in de Jong-van Straten [2].

The decoration l associated with a decorated curve (C, l) can be thought of as the combination of unique subschemes, each having length l_i, supported on the preimage of the origin 0 on the normalization of C_i.

DEFINITION 2.6 (de Jong-van Straten [2, Definition 4.2]). Suppose (C, l) is a decorated curve associated with a sandwiched surface singularity (X, p). Then, a *picture deformation* $(\mathcal{C}, \mathcal{L})$ of (C, l) over a small disk Δ centered at the origin 0 consists of the following:

(1) A δ-constant deformation $\mathcal{C} \to \Delta$ of C, meaning that $\delta(C_t)$ is constant for all $t \in \Delta$, where C_t is a fiber over t.
(2) A flat deformation $\mathcal{L} \subset \mathcal{C}$ over Δ of the scheme l.
(3) $\mathcal{M} \subset \mathcal{L}$, where the relative total multiplicity scheme $\mathcal{M}$ of $\mathcal{C} \to \Delta$ is defined as the closure $\bigcup_{t \in \Delta \setminus 0} m(C_t)$.
(4) For generic $t \in T \setminus 0$ the divisor l_t on $\widetilde{C}_t$ is reduced.

Proposition 2.7 (de Jong-van Straten [2, Theorem 4.4]). *For any one-parameter smoothing of X, there corresponds to a picture deformation of its decorated curve (C, l), and vice versa.*

2.3. Incidence matrices

The combinatorial aspects of picture deformations for (C, l) can be captured using specific matrices. Let's consider a picture deformation $(\mathcal{C}, \mathcal{L})$ of (C, l). Assuming that C is composed of components $C = \cup_{i=1}^{s} C_i$ and that $C_t = \cup_{i=1}^{s} C_{i,t}$ for the deformed curve C_t, we denote by $P_1, \ldots, P_n$ the images in C_t of the points in the support of l_t.

Definition 2.8 (de Jong-van Straten [2, p. 483]). The *incidence matrix* of a picture deformation $(\mathcal{C}, \mathcal{L})$ is represented as $I(\mathcal{C}, \mathcal{L}) \in M_{s,n}(\mathbb{Z})$, where $M_{s,n}(\mathbb{Z})$ is the set of $s \times n$ matrices whose entries are integers. In this matrix, the entry at position (i, j) is equal to the multiplicity of P_j as a point on $C_{i,t}$ for $t \neq 0$.

Lemma 2.9. *The incidence matrices that corresponds to the rational homology disk smoothing of $W_{p,q,r}$ is given as follows:*

$$M = \begin{array}{c} \\ A(I) \\ \\ B(I) \\ \\ C(I) \end{array} \left[\begin{array}{ccc|ccc|ccc} \multicolumn{3}{c|}{\overbrace{\quad\quad}^{q+2}} & \multicolumn{3}{c|}{\overbrace{\quad\quad}^{p+2}} & \multicolumn{3}{c}{\overbrace{\quad\quad}^{r+2}} \\ 1 & \cdots & 1 & 1 & & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & & 1 & 0 & \cdots & 0 \\ \hline 1 & & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & & 1 \end{array} \right]$$

In addition, if $p = q = r$, then we have one more incidence matrix:

$$M' = \begin{array}{c} A(I) \\ \\ B(I) \\ \\ C(I) \end{array} \left[\begin{array}{ccc|ccc|ccc} 1 & \cdots & 1 & 1 & & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & & 1 \\ \hline 1 & & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{array} \right]$$

Proof. See Park–Shin [4, §16.3]. □

3. $\mathbb{Q}$-Gorensteinness

de Jong and van Straten [2] provided a straightforward criterion for determining whether a one-parameter smoothing of a sandwiched surface singularity is $\mathbb{Q}$-Gorenstein.

PROPOSITION 3.1 (de Jong and van Straten [2, Corollary 5.12]). *A one-parameter smoothing $\mathcal{X} \to \Delta$ is $\mathbb{Q}$-Gorenstein if and only if $(1, 1, \ldots, 1)$ is a rational linear combination of the rows of the corresponding incidence matrix.*

Proof of Theorem 1.1. We can verify that the matrix M has full rank, meaning that there is always a solution to the matrix equation $M^T Y = (1, \ldots, 1)^T$. It's important to note that the determinants of M^T and its submatrices are always rational numbers, implying the existence of a rational solution Y. Consequently, we can express $(1, 1, \ldots, 1)$ as a rational linear combination of the rows of M. Similarly, we can prove the criterion to the matrix M'. □

References

[1] M. Bhupal, A. Stipsicz, *Weighted homogeneous singularities and rational homology disk smoothings*, Amer. J. Math., **133** (2011), no. 5, 1259–1297.

[2] T. de Jong, D. van Straten, *Deformation theory of sandwiched surface singularities*, Duke Math. J., **95** (1998), no. 3, 451–522.

[3] J. Kollár, N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math., **91** (1988), no. 2, 299–338.

[4] H. Park, D. Shin, *Deformations of sandwiched surface singularities and the minimal model program*, arXiv:2205.11165.

[5] M. Spivakovsky, *Sandwiched surface singularities and desingularization of surfaces by normalized Nash transformations*, Ann. Math., **131** (1990), 411–491.

[6] J. Wahl, *Smoothings of normal surface singularities*, Topology **20** (1981), no. 3, 219–246.

[7] J. Wahl, *Log-terminal Smoothings of Graded Normal Surface Singularities*, Michigan Math. J., **62** (2013), 475–489.

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